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Two-sided symmetric is NP-hard

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We will consider the problem of maximizing two-sided Nash welfare under symmetric valuations. We have hardness results for both the general problem of maximizing two-sided Nash welfare (when valuations are not necessarily symmetric) and for the problem of maximizing one-sided Nash welfare (on the firm-side). Nevertheless, there are no results for the two-sided symmetric case. We know that this problem is **different** from the one-sided case; i.e. there does not exist any algorithm that provides a constant-factor approximation to both problems. We prove that, as expected, the two-sided symmetric case is NP-hard.

We will consider constant capacities; specifically, $\forall i, c_i = 2$.

Definition (`RAINBOW-PERFECT-MATCHING-3`)**.** Let $G = (X \cup Y, E)$ be a bipartite multigraph with edges $e \in E$ between sets $X, Y : |X| = |Y| = r$. Moreover, the set of edges is partitioned into r color classes. Between each pair of vertices, there may be multiple edges of different colors (but at most one each of a single color). The problem is to decide whether or not there exists a perfect matching that uses r edges; one each of every color. Additionally, each vertex has degree 3 and there are exactly 3 edges of every color.

Theorem (Jain & Vaish)**.** `RAINBOW-PERFECT-MATCHING-3` is NP-complete.

To show the reduction, we will consider the decision version of the two-sided symmetric problem, defined as follows:

Definition (Two-sided symmetric decision problem)**.** The input consists of sets of firms $\mathcal{F}, |\mathcal{F}| = n$ and workers $\mathcal{W}, |\mathcal{W}| = m$, symmetric valuations v_{ij}, capacities for firms c_i, and a threshold $\theta > 0$. The problem is to decide whether or not there exists a matching such that the two-sided Nash welfare $\geq \theta$. We will call this decision problem `TWO-SIDED-SYMMETRIC`.

Theorem. `TWO-SIDED-SYMMETRIC` $\leq_p$ `RAINBOW-PERFECT-MATCHING-3`.

Proof. (Modified from Vaish & Jain). We will prove this by transforming any yes-instance of `RAINBOW-PERFECT-MATCHING-3` to a yes-instance of `TWO-SIDED-SYMMETRIC` and vice versa.

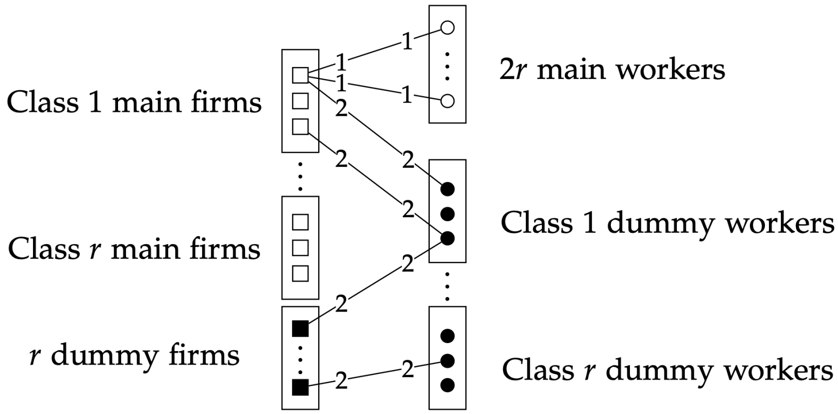

Figure 1: *Yes-instance for `TWO-SIDED-SYMMETRIC`.*

($\Rightarrow$) Assume that $G = (X \cup Y, E)$ is a yes-instance of `RAINBOW-PERFECT-MATCHING-3`. We will construct an instance of `TWO-SIDED-SYMMETRIC` as follows (in poly-time):

There are $n = 4r$ firms and $m = 5r$ workers, given that $|X| = |Y| = r$. For each of the $2r$ vertices in $X \cup Y$, we will have a **main worker** corresponding to that vertex. For each edge $e = (u, v) \in E$, we will have a **main firm** corresponding to that edge. Additionally, for edge e, we define $v_{ij} = 1, v_{ik} = 1$, where main firm i corresponds to the edge $e = (u, v)$ and main workers j, k correspond to vertices u, v respectively. Since the edge set E consists of exactly 3 edges of each color, there are exactly $3r$ main firms, 3 corresponding to each of the r colors. Thus, we can partition the set of main firms into r classes as in Figure 1. We will also define $3r$ **dummy workers**, one corresponding to each main firm (similarly, we can think of dummy workers partitioned into r classes of 3). We define $v_{ab} = 2$ where b is the dummy worker corresponding to each main firm a. Finally, we will define r **dummy firms**, one corresponding to each class of dummy workers. Then, $v_{cp} = v_{cq} = v_{cr} = 2$ for dummy firm c and dummy workers p, q, r that belong to some class $\in [r]$. We define the threshold $\theta = 2 \cdot 2^{3/5}$.

Since G is a yes-instance, there must exist some perfect matching μ that uses one edge of each color. Now, we will define a matching x in our instance. For each edge in μ, we will match the main firm corresponding to that edge to the two main workers corresponding to that edge. At the end of this process, there are $2r$ unmatched main firms, two in every class. For each unmatched firm, match that firm to the dummy worker corresponding to it. Match the remainder of the r dummy workers (one in each class) to the dummy firm corresponding to that class. It is easy to verify that all firms and workers are matched. Each firm and dummy worker receives utility 2; each main worker receives utility 1. Thus, the two-sided Nash welfare is $(2^{4r})^{\frac{1}{4r}} \cdot (2^{3r} \cdot 1^{2r})^{\frac{1}{5r}} = 2 \cdot 2^{3/5} \geq \theta$. Thus, our constructed instance is indeed a yes-instance of `TWO-SIDED-SYMMETRIC`.

($\Leftarrow$) The reverse direction is equivalent to showing we can map (in poly-time) no-instances of `RAINBOW-PERFECT-MATCHING-3` to no-instances of `TWO-SIDED-SYMMETRIC`. Let G be a no-instance of `RAINBOW-PERFECT-MATCHING-3`. The goal is to construct, in poly-time, a no-instance of the `TWO-SIDED-SYMMETRIC` problem. Our construction is the exact same as in the forward direction (refer to Figure 1). All that remains is to prove that the construction is a no-instance of `TWO-SIDED-SYMMETRIC`.

Assume for sake of contradiction that there exists a matching x in the construction that achieves a two-sided Nash welfare $\geq \theta = 2 \cdot 2^{3/5}$. This implies that all firms and workers must be matched (since otherwise, the Nash welfare would be 0). Thus, all dummy workers must receive utilities of 2 each, and all main workers 1 each. This implies that the geometric mean of all firm utilities must be ≥ 2. Now, consider the total utility derived by all firms. This must be equal to the total utility derived by all workers (from symmetry) which we know to be $1 \cdot 2r + 2 \cdot 3r = 8r$. Since there are $4r$ firms, the arithmetic mean of all firm utilities is exactly $8r/4r = 2$. From the arithmetic mean-geometric mean inequality, we know that the arithmetic mean and geometric mean must both exactly be 2. It is well known that equality occurs iff all numbers are equal $\implies$ all firms derive utility 2 each. Since all dummy firms are matched to exactly one dummy worker, there must be 2 dummy workers from each color class whom are matched with their corresponding main firm. This leaves 1 main firm from each color class who is matched with both corresponding main workers. Considering the edges corresponding to all of these unmatched main firms implies a rainbow perfect matching μ in G, a contradiction. This completes our proof.

We have shown `TWO-SIDED-SYMMETRIC` $\leq_p$ `RAINBOW-PERFECT-MATCHING-3`. □

Combining both theorems, we have shown that the decision version of our optimization problem is NP-complete, which implies that the optimization problem of maximizing two-sided Nash welfare under symmetric valuations is NP-hard (since the decision problem clearly reduces to the optimization problem).