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Two-sided symmetric $\neq$ one-sided

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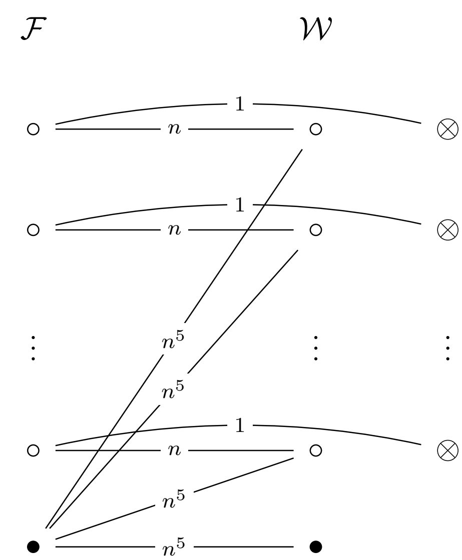
We simplify the problem of maximizing two-sided Nash welfare by assuming $\forall i, j\ v_{ij}^{\mathcal{F}} = v_{ij}^{\mathcal{W}}$ (symmetric valuations for firms and workers). Recall that we assume that all workers have unit capacity and $\forall i$, firm i has capacity $c_i \geq 1$. Given this simplification, we ask (informally): is solving the problem on one-side (for the firms) good enough for two-sided welfare? The motivation for asking this question is that, if this question were answered in the positive, we could apply any constant-factor approximation algorithm (e.g. by Cole & Gkatzelis) for the one-sided problem to the two-sided problem. We answer this question in the **negative**.

Theorem. There exists an instance of n firms and $2n-1$ workers with valuations $v_{ij}^{\mathcal{F}} = v_{ij}^{\mathcal{W}} = v_{ij}$ and firm capacities c_i such that $\forall$ allocations x, if x attains a constant-factor approximation to the maximum one-sided welfare, then x does not attain a constant factor approximation to the maximum two-sided welfare.

Proof. Consider the following instance with n firms and $2n-1$ workers. We only indicate edges that have non-zero valuation. For all edges, valuations are symmetric.



Assume that the last firm (indicated by $\bullet$) matches with $c \in [n]$ workers.

One-sided welfare. With respect to the instance above, the one-sided utility (for the firms) is the following (when the last firm matches with c workers):

$$w_1(c) = (n+1)^{\frac{n-c}{n}} \cdot (c \cdot n^5)^{\frac{1}{n}}$$

The one-sided welfare (w_1) is fully determined by c since every worker labelled $\circ$ can either be matched with its corresponding firm $\circ$ or with the last firm $\bullet$. Every worker labelled $\otimes$ must be matched with the corresponding $\circ$ firm (it has only one non-zero valuation). It is easy to see that $c = 1$ maximizes the function above $\forall$ suitable n. For suitable n, we see that the function w_1 is monotonically decreasing in c. Thus:

$$\mathrm{OPT}_1 = \max_{c \in [n]} \{w_1(c)\} = w_1(1) = (n+1)^{\frac{n-1}{n}} \cdot (n^5)^{\frac{1}{n}}$$

When $c = \Theta(n)$, it is not possible to have a constant-factor approximation to OPT_1.

Lemma. When $c = \Theta(n)$, $w_1(c) \in o(\mathrm{OPT}_1)$ (treating $w_1(c)$ and OPT_1 as functions of n).

Proof. Consider limit definition of $o(\cdot)$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{w_1(c = \Theta(n))}{\mathrm{OPT}_1(n)} &= \lim_{n\to\infty} (n+1)^{\frac{1-c}{n}} \cdot c^{\frac{1}{n}} \\
&= \lim_{n\to\infty} (n+1)^{\frac{1-\Theta(n)}{n}} \cdot \lim_{n\to\infty} \Theta(n)^{\frac{1}{n}} \\
&= 0 \cdot 1 = 0
\end{aligned}$$

Thus, we cannot achieve a constant-factor approximation to the maximum one-sided welfare when c is on the order of n. $\square$

Next, we will show that when $c = o(n)$, we cannot achieve a constant factor approximation to the maximum two-sided welfare.

Two-sided welfare. With respect to the provided instance, the two-sided utility is the following (when the last firm matches with c workers):

$$w_2(c) = w_1(c) \cdot (n)^{\frac{5c}{2n-1}} \cdot (n)^{\frac{n-c}{2n-1}} = (n+1)^{\frac{n-c}{n}} \cdot (c \cdot n^5)^{\frac{1}{n}} \cdot (n)^{\frac{n+4c}{2n-1}}$$

Since the growth rate of the second term (expressing the worker valuations) is so large, the function $w_2(c)$ is monotonically increasing in c for suitable n. Thus, $c = n$ maximizes the function above $\forall$ suitable n. Thus:

$$\mathrm{OPT}_2 = \max_{c \in [n]} \{w_2(c)\} = w_2(n) = (n^6)^{\frac{1}{n}} \cdot (n)^{\frac{5n}{2n-1}} \geq n^{5/2}$$

Finally, we will complete our proof with the following lemma:

Lemma. When $c = o(n)$, $w_2(c) \in o(\mathrm{OPT}_2)$ (treating $w_2(c)$ and OPT_2 as functions of n).

Proof. Consider limit definition of $o(\cdot)$:

$$\begin{aligned} \lim_{n \to \infty} \frac{w_2(c = o(n))}{\mathrm{OPT}_2(n)} &\leq \lim_{n \to \infty} \frac{(n+1)^{\frac{n-o(n)}{n}} \cdot (o(n) \cdot n^5)^{\frac{1}{n}} \cdot (n)^{\frac{n+4o(n)}{2n-1}}}{n^{5/2}} \\ &= 0 \quad (n^{5/2} \text{ grows faster than all numerator terms}). \end{aligned}$$

Thus, we cannot achieve a constant-factor approximation to the maximum two-sided welfare when c is not on the order of n (sub-linear). □

Thus, we have shown that it is impossible for any allocation x to attain a constant factor approximation to both maximum one-sided and two-sided welfare on all instances, even when valuations are symmetric. □

Remark. Using the same strategy, it is possible to construct an instance (under symmetric valuations) where a constant-factor approximation to the one-sided welfare **for workers** does not give a constant-factor approximation to two-sided welfare. I believe that it is also possible to construct an instance where a constant-factor approximation to either side of the problem does not give a constant-factor approximation to the two-sided welfare (this is yet to be proved formally). These instances illustrate that the two-sided problem (under symmetric valuations) is certainly different from the one-sided problem; that is, we cannot directly apply one-sided algorithms, even when valuations are symmetric.